Electrostriction and magnetostriction Casimir force when $\epsilon\mu=1$

1982 J. Phys. A: Math. Gen. 15 L369

(http://iopscience.iop.org/0305-4470/15/7/008)

J. Phys. A: Math. Gen. 15 (1982) L369–L373. Printed in Great Britain

LETTER TO THE EDITOR

Electrostriction and magnetostriction Casimir force when $\varepsilon\mu = 1$

I Brevik

Luftkrigsskolen, N-7000 Trondheim, Norway

Received 30 March 1982

Abstract. The combined electrostriction and magnetostriction Casimir force on a compact sphere placed in a vacuum is calculated, provided that the medium satisfies the relationship $\varepsilon\mu = 1$, ε being the permittivity and μ the permeability. This special class of media is known to possess attractive properties (Brevik and Kolbenstvedt 1982a, b): cut-off terms that otherwise represent a difficulty in Casimir calculations for ordinary non-magnetic media (Milton 1980, Brevik 1982a) simply cancel out in the surface force expression. This note examines the question whether similar cancellations occur in the *striction* force when the magnetostriction part is included. The answer is negative.

The current interest in calculations of the Casimir effect for spherical geometry is motivated by its importance for electrodynamics as well as for quantum chromodynamics. Whereas the idea of picturing the semiclassical electron as a conducting shell stabilised by the zero-point fluctuations is due to Casimir (1956), it was only after the advent of specific calculations by Boyer (1968), Balian and Duplantier (1978), Milton *et al* (1978) and others that it became clear that the surface force is acting *outwards*, contrary to Casimir's original suggestion. Milton *et al* (1978) calculated the surface force density to great accuracy:

$$F_0 = 0.09235/(8\pi a^4), \tag{1}$$

a being the radius. The remarkable property here is that terms containing the (temporal) cut-off parameter cancel out.

One interesting generalisation of the semiclassical electron model is to consider a dielectric compact sphere instead of a thin conducting shell. This case was first worked out by Milton (1980) for non-magnetic media. The result was that cut-off terms were remaining even after the subtraction of suitable 'contact' terms. The present author went one step further and considered the electrostrictive contribution to the Casimir force on the sphere, still assuming a non-magnetic medium (Brevik 1982a). The motivation for this undertaking was the great importance of the electrostriction effect in general for the distribution of hydrostatic pressure in a fluid in classical electromagnetism (cf, for instance, Brevik 1982b, 1979). Similarly as in Milton's case, cut-off terms were found to survive, thus indicating that the underlying physical model is incomplete in some way.

It is rather remarkable that these cut-off problems go away if the medium is permitted to possess, apart from a permittivity ε, also a permeability μ, such that the relationship

$$\varepsilon\mu = 1 \tag{2}$$

0305-4470/82/070369+05$02.00

is satisfied. In this case, similar cut-off cancellations occur as in the standard case of a conducting shell. If the compact sphere is surrounded by a vacuum, the Casimir surface force density on it is (Brevik and Kolbenstvedt 1982b; a brief note in 1982a)

$$F_{\text{surf}} = F_0 \left(\frac{\mu - 1}{\mu + 1}\right)^2 \left(1 + 0.311 \frac{\mu}{(\mu + 1)^2}\right) \tag{3}$$

(where we have used the symbol F_{surf} instead of F to distinguish it from the striction force). The condition (2) is formally exactly the condition made in QCD to ensure that the gluons propagate with the velocity of light.

The question to which we address ourselves here is the following: is the cut-off independence in the force a property that is shared also by the *striction* force, if we allow for a magnetostrictive as well as an electrostrictive contribution, and assume a medium such that (2) holds? From the outset it might appear as a realistic possibility that an equivalent treatment of the electric and magnetic fields, which is implied by (2), would be sufficient to establish the nice cut-off independence property for the striction force also.

The result of the explicit calculation below is that the cut-off terms do *not* go away. There is, however, one advantage in handling the electrostrictive and magnetostrictive contributions at the same time: the contact term, that is to be subtracted off in the formalism in order to obtain the physical force, becomes equal to zero.

The calculation is conveniently carried out using Green function methods. We may start from the expression for the electromagnetic force density in isotropic matter (cf, for instance, Landau and Lifshitz 1960),

$$f = -\tfrac{1}{2}E^2\nabla\varepsilon - \tfrac{1}{2}H^2\nabla\mu + \nabla[\tfrac{1}{2}E^2\rho\, \mathrm{d}\varepsilon/\mathrm{d}\rho] + \nabla[\tfrac{1}{2}H^2\rho\, \mathrm{d}\mu/\mathrm{d}\rho], \tag{4}$$

in which the two first terms are non-vanishing only in the boundary layer at the surface, and are responsible for the surface force density given in (3) when (2) is valid. Of interest for us are the two striction terms in (4). Here ρ is the matter density (the expression applies strictly speaking to an isotropic fluid), and we adopt a non-polar model of the medium so that ε and μ become temperature independent. In virtue of (2) we write the striction force density, i.e. the sum of the electrostriction and magnetostriction terms, as

$$f_{\text{str}} = \nabla[\tfrac{1}{2}(E^2 - B^2)\rho\, \mathrm{d}\varepsilon/\mathrm{d}\rho]. \tag{5}$$

This expression in general is different from zero both in the interior of the sphere (because of the varying fields) and in the boundary region at the surface where ε changes abruptly with position.

To find the action from (5) on the sphere we integrate, as in Brevik (1982a), the radial component over a narrow sector of the sphere covering a solid angle element $\mathrm{d}\Omega$. The result is written as $a^2\,\mathrm{d}\Omega F_{\text{str}}$, so that F_{str} has the dimension of a surface force density. By means of a partial integration over r we obtain

$$F_{\text{str}} = -\frac{1}{a^2}\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}\rho}\int_0^a (E^2 - B^2)r\, \mathrm{d}r. \tag{6}$$

Now adopting the Clausius–Mossotti relation in the evaluation of $\mathrm{d}\varepsilon/\mathrm{d}\rho$, and inserting the effective product of two electric or two magnetic fields from Brevik and Kolbenstvedt (1982b), we obtain

$$F_{\rm str}=\frac{(\mu-1)(2\mu+1)}{3\mathrm{i}\mu a^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega\tau}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}\int_0^a r\,\mathrm{d}r\left(\omega^2-\frac{l(l+1)}{r^2}\right.$$
$$\left.-\frac{1}{r^2}\frac{\partial}{\partial r}r\frac{\partial}{\partial r'}r'\right)[F_l(r,r')-G_l(r,r')]\bigg|_{r=r'}. \tag{7}$$

Here $\tau=t-t'$ measures the temporal separation of the two space–time points x and x'. F_l and G_l are the scalar Green functions ($k=|\omega|$):

$$F_l,G_l=\mathrm{i}kj_l(kr_<)[h_l^{(1)}(kr_>)-A_{F,G}j_l(kr_>)], \tag{8}$$

where ($z=ka$)

$$A_F=\frac{\tilde{D}_l(z)}{D_l(z)}\quad A_G=(\mu-1)\frac{e_l(z)e_l'(z)}{D_l(z)}, \tag{9}$$

$$D_l(z)=\mu s_l(z)e_l'(z)-s_l'(z)e_l(z), \tag{10}$$

$$\tilde{D}_l(z)=\mu s_l'(z)e_l(z)-s_l(z)e_l'(z), \tag{11}$$

$s_l=zj_l$ and $e_l=zh_l^{(1)}$ being the Riccati–Bessel functions. Now defining $y=\omega a$, $z=|y|$, $\delta=\tau/a$, we may after some algebra write (7) in the form

$$F_{\rm str}=\frac{(\mu-1)^2(2\mu+1)(\mu+1)}{6\mathrm{i}\mu a^4}\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{2\pi}\mathrm{e}^{-\mathrm{i}\delta y}z\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}\frac{e_l(z)e_l'(z)}{D_l(z)\tilde{D}_l(z)}$$
$$\times\int_0^z\frac{\mathrm{d}q}{q}[s_l^2(q)]''. \tag{12}$$

This is our general result.

One important observation can be made at this stage: we do not have to subtract off a contact term in order to obtain the physical force. The contact term is constructed solely from one scalar Green function,

$$F_l=G_l=F_l^{(0)}=\mathrm{i}kj_l(kr_<)h_l^{(1)}(kr_>), \tag{13}$$

and since from (7) it is clear that only the difference between F_l and G_l appears, the contact term must be expected to give the physical force directly, as following from our model.

To evaluate the two integrals and the sum in (12) in the general case would be quite difficult. Some simplification is achieved if we restrict ourselves to the case of a dilute medium, i.e. $|\mu-1|\ll 1$. We then obtain to first order in $(\mu-1)$, employing the Wronskian $W\{s_l,e_l\}=\mathrm{i}$ (in conventional normalisation), $D_l\tilde{D}_l\to\mu$. Thus (12) yields, to lowest order,

$$F_{\rm str}=\frac{(\mu-1)^2}{\mathrm{i}a^4}\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{2\pi}\mathrm{e}^{-\mathrm{i}\delta y}z\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}e_l(z)e_l'(z)\int_0^z\frac{\mathrm{d}q}{q}[s_l^2(q)]'', \tag{14}$$

where we recover in front the proportionality to the square of the susceptibility which is so typical for dilute media.

There is still a difficulty here in calculating the integral over q for arbitrary z. (For large values of z it would be easy to calculate this integral to a good approximation: by means of two partial integrations we might write the integral as terms containing $s_l(z)$

plus a remainder integral which is of the Weber–Shafheitlin type if the upper limit (z) is replaceable by infinity.) Our main interest here is not however the explicit calculation of (14) but rather to answer the following question: is the expression independent of cut-off, and finite? In fact, it is *not*. It is cut-off divergent even for the lowest mode separately, $l = 1$. It is relatively easy to show this, making use of (Abramowitz and Stegun 1964)

$$s_1(z) = z^{-1} \sin z - \cos z, \qquad e_1(z) = -(1 + iz^{-1})\, e^{iz}. \tag{15}$$

We obtain

$$\int_0^z \frac{dq}{q} [s_1^2(q)]'' = \frac{1}{2} - \frac{3}{4z^4} + \left(-\frac{1}{z} + \frac{3}{2z^3}\right) \sin 2z + \left(-\frac{3}{2z^2} + \frac{3}{4z^4}\right) \cos 2z. \tag{16}$$

Performing now a complex frequency rotation (cf, for instance, Milton *et al* 1978)

$$\omega \to ik_4, \qquad k \to i|k_4|, \qquad \tau \to i\tau_4, \tag{17}$$

we are in the evaluation of the y integral in (14) confronted with a number of exponential integrals and cosine integrals that can be calculated separately making the same effective substitutions as in equations (5.8)–(5.11) in Brevik (1982a). The final result for the $l = 1$ contribution becomes

$$F_{\text{str}}(l=1) = \frac{3(\mu-1)^2}{10\pi^2 a^4} [-\tfrac{5}{8} \ln \delta + \tfrac{5}{4}\pi\delta(\delta) - 1 + 2 \ln 2], \tag{18}$$

where $\delta(\delta)$ means the delta function of δ.

The $l = 1$ contribution to the striction force is thus cut-off divergent. The full striction force, obtained by summing over all l, will also have to be divergent. Thus the behaviour is in this respect analogous to that found for a non-magnetic medium. The remaining finite terms in (18) yield a repulsive contribution. It ought to be observed, however, that we cannot draw any conclusion as to the repulsiveness of the full force when summed over all l; the sign may change during the summation. The striction force is a result of an integration over r out to $r = a$, and so we must expect that the higher modes may be important. (What makes the $l = 1$ contribution so important in describing gluon and quark condensates in QCD, is that the field values near the *centre* of the bag seem to be so typical; see Milton (1981).)

Our main conclusion thus becomes the following. The nice cut-off independence property shown by the ordinary surface force (the force corresponding to the two first terms in (4)) is not shared by the striction force. This indicates that as far as the semiclassical electron model, or the bag QCD model, are concerned, the surface force is of greater fundamental importance than the striction force. It is however worth noticing that when the electrostriction and magnetostriction forces are combined, there is no longer any need for subtracting off contact terms.

References

Abramowitz M and Stegun I A (ed) 1964 *Handbook of Mathematical Functions* (Washington, DC: National Bureau of Standards) (Reprinted by Dover, New York)
Balian R and Duplantier B 1978 *Ann. Phys., NY* **112** 165–208
Boyer T H 1968 *Phys. Rev.* **174** 1764–76

Brevik I 1979 *Phys. Rep.* **52** 133–201
—— 1982a *Ann. Phys., NY* **138** 36–52
—— 1982b *Can. J. Phys.* **60** in press
Brevik I and Kolbenstvedt H 1982a *Phys. Rev.* D **25** 1731–4
—— 1982b *The Casimir Effect in a Solid Ball when $\varepsilon\mu = 1$, Det Fysiske Seminar i Trondheim* No 1
Casimir H B G 1956 *Physica* **19** 846–9
Landau L D and Lifshitz E M 1960 *Electrodynamics of Continuous Media* (Oxford: Pergamon)
Milton K A 1980 *Ann. Phys., NY* **127** 49–61
—— 1981 *Phys. Lett.* **104B** 49–54
Milton K A, DeRaad Jr L L and Schwinger J 1978 *Ann. Phys., NY* **115** 388–403